

Observable Long-Run Ambiguity and Long-Run Risk

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Abstract

I propose an equilibrium model where the investor is uncertain about which data generating process drives the predictable (long-run risk) component in consumption growth and inflation. This type of observable Knightian uncertainty makes the equilibrium marginal utility to be heteroscedastic, even if consumption and inflation are conditionally homoscedastic. I find that the model explains how model misspecification doubts about the predictable components feeds into time-varying bond premiums and option implied volatilities that are unspanned by the yield curve. The equilibrium explains the volatility skew and 50% of variations in the equity-index options and T-Note futures options.

Keywords: Bond premium, VIX, Unspanned volatility, yield curve, Knightian uncertainty, implied volatility skew

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1. Introduction

Does marginal utility account for macro uncertainty? The bond market is the natural laboratory for this question, because bond yields and bond volatilities are deterministic transformations of the investor's marginal utility.

The answer is yes, it does. Current models with Epstein and Zin (1989) or Campbell and Cochrane (1999) preferences solve several asset pricing puzzles through accounting for macro uncertainty via exogenously specified stochastic volatility or volatility-of-volatility processes in GDP growth or inflation.² Usually, these volatilities must be filtered from financial data, which increases the complexity of the model implementation.

In this paper, I analyze whether a complementary modeling of uncertainty, with only observable macro variables, helps to explain dynamics of marginal utility. I model macro uncertainty as macro ambiguity in the sense of Knight (1921), Hansen and Sargent (2005), Chen and Epstein (2002) and Epstein and Schneider (2003). One generalization to the successful Bansal

²Recent accounts for long-run risk models are Bansal and Yaron (2004), Bansal et al. (2007), Bollerslev et al. (2009b), Bansal and Shaliastovich (2006), Bansal and Shaliastovich (2009), Bansal and Shaliastovich (2010b), Piazzesi and Schneider (2006), Piazzesi and Schneider (2010), Drechsler and Yaron (2010) among others. Bekaert et al. (2009) find evidence that Campbell and Cochrane (1999) preferences and stochastic volatility in consumption and inflation are able to account for important bond and stock phenomena. Drechsler and Yaron (2010) show that jumps in volatility and in the predictable long-run risk component are important to quantitatively match the size, volatility and high skewness of the variance premium.

and Yaron (2004) long-run risk literature is that the investor in my model is uncertain about which data generating process drives the predictable (long-run risk) component in consumption growth and inflation.³ This uncertainty makes the equilibrium marginal utility to be heteroscedastic, even if consumption and inflation are conditionally homoscedastic. My findings indicate that accounting for model misspecification of the predictable long-run risk components explains why bond premiums and option implied volatilities are unspanned by the yield curve (Cochrane and Piazzesi (2005), Ludvigson and Ng (2009), and Duffee (2010)).

Patton and Timmermann (2010) provide empirical support for the hypothesis of model uncertainty.⁴ I build on that empirical evidence and suggest an analytically tractable, and empirically easy to estimate, framework that allows to capture Knightian uncertainty about GDP growth and inflation. State-of-the art equilibrium models with Knight (1921) uncertainty focus either on GDP ambiguity or on inflation ambiguity.⁵ My modeling

³Bansal and Shaliastovich (2009) show that long-run components in consumption growth and inflation are necessary to model bond markets.

⁴Early accounts for implementing this into models of ambiguity aversion are Anderson et al. (2009) and Ulrich (2010). Buraschi et al. (2009), David and Veronesi (2009), Buraschi and Jiltsov (2006), David and Veronesi (2002), among others, use dispersion in forecasts as a measure for heterogeneity in beliefs.

⁵Prominent Knightian uncertainty frameworks, such as the equity model of Cagetti et al. (2002), or the confidence risk model of Bansal and Shaliastovich (2010a), as well as the real yield curve models of Gagliardini et al. (2009) and Kleshechelski and Vincent (2009) focus on GDP ambiguity as the main uncertainty driver in the agent's marginal utility. Note that Bansal and Shaliastovich (2010a) do not assume min-max preferences but their confidence measure for GDP ambiguity is consistent with the logic of a set of

framework allows easily to account for both types of uncertainty.

In order to focus on the Knightian uncertainty channel, I focus on a Lucas (1978) type economy with a log utility agent and conditionally homoscedastic inflation and consumption processes. The same framework without Knightian uncertainty is known to generate all types of asset pricing puzzles. But the clear advantage of that framework is that if adding Knightian uncertainty helps to explain part of these asset pricing puzzles, we can be sure it is because of the proposed uncertainty channel. The tractability of my framework allows an easy extension to more complex macro processes and different preference specifications.⁶ Despite the simplicity of the framework, the research questions it addresses are complex.

First, does GDP ambiguity or inflation ambiguity dominate variations in marginal utility? I document that data from the Survey of Professional Forecasters reveals a higher amount of GDP ambiguity. This supports Cagetti et al. (2002), Bansal and Shaliastovich (2010a) and Gagliardini et al. (2009) who build equilibrium models with GDP ambiguity. On the other hand, the bond market implications of the model reveal that inflation ambiguity is of bigger concern to the investor. This supports Ulrich (2010) who finds evi-

models. On the other hand, Ulrich (2010) explains the nominal term spread in Treasury yields through aversion against inflation uncertainty.

⁶Drechsler (2009) shows how to extend a Knightian uncertainty set-up to stochastic volatility, disaster risk and Epstein and Zin (1989) preferences.

dence that inflation ambiguity can account for the nominal term spread in U.S. Treasuries.⁷

Second, my analysis concludes that incorporating macro ambiguity into an otherwise classical Lucas (1978) model, makes a significant step towards explaining several daunting bond pricing puzzles. The endogenously determined equilibrium bond premiums are stochastic, although consumption and inflation volatility are conditionally homoscedastic. A variance decomposition of the bond premiums reveals that an econometrician would conclude that the endogenous bond premiums are unspanned by the yield curve. This underlines the importance of the equilibrium channel, because Cochrane and Piazzesi (2005), Ludvigson and Ng (2009), and Duffee (2010) find that bond premium factors are partly unspanned by the yield curve.

The analysis also finds that observable macro ambiguity helps to generate time-varying volatility in bond yields. Moreover, the model implied volatility of bond options exhibits a pronounced Black (1976) implied volatility skew. That skew is entirely driven by the observable amount of long-run ambiguity and appears to an econometrician to be unspanned, as well. This is another promising feature of the proposed mechanism, because Collin-Dufresne and Goldstein (2002) and Heidari and Wu (2003) find evidence that the volatil-

⁷Buraschi and Jiltsov (2005), Ang et al. (2008), Piazzesi and Schneider (2006), Gürkaynak et al. (2005) find further evidence for the importance of inflation risk.

ity of bond yields and option implied volatilities are unspanned by the yield curve. The proposed ambiguity channel is different to Bansal and Shaliastovich (2009) who generate time-varying volatility in bond yields through Epstein and Zin (1989) preferences and stochastic volatility in consumption and inflation. It is also different to Bekaert et al. (2009) who assume Campbell and Cochrane (1999) preferences with stochastic volatility in consumption growth and inflation.

The endogenous uncertainty premiums for long-run ambiguity drive the Black (1976) implied volatility of bond options. A contemporaneous regression reveals that the model explains 50% of the variations in the CBOE VIX and 50% of the volatility of bond options. This high explanatory power for options supports the usefulness of the proposed model mechanism.

The paper is structured as follows. Section 2 describes the model and derives equilibrium bond yields, bond premia, yield volatility and specifies the interest rate option contract of interest. I estimate the model in section 3 with bond yield, bond variance and macro data. Results are explained and implications for bond premia and bond volatility are visualized in section 3, as well. Section 4 quantifies the detection error. Section 5 concludes. The appendix contains derivations and proofs.

2. Model

I work with an endowment economy. Time is continuous and varies over $t \in [0, \dots, \infty)$. I assume a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, Q^0)$, where Q^0 stands for the probability distribution of the investor's most trustworthy macro model (reference model). I denote expectations under Q^0 as $E[\cdot]$ instead of $E^{Q^0}[\cdot]$. The solution to the dynamic Gilboa and Schmeidler (1989) type min-max problem determines endogenously the worst-case probability measure Q^h . For simplicity, all Brownian motions are pairwise orthogonal.

2.1. Assumptions on the Most Trustworthy Economy

Realized growth of the investor's endowment follows a homoscedastic process with a time-varying trend growth rate z

$$d \ln c_t = (c_0 + z_t)dt + \sigma_c dW_t^c, \quad (1)$$

with $c_0 > 0$ and $\sigma_c > 0$. The mean zero trend growth rate, z , follows a continuous-time AR(1) process

$$dz_t = \kappa_z z_t dt + \sigma_{1z} dW_t^r + \sigma_{2z} dW_t^w \quad (2)$$

with $\kappa_z < 0$, $\sigma_{1z} > 0$, and $\sigma_{2z} < 0$. I call z the predictable trend component in expected consumption growth.

The exogenous process for inflation, $d \ln p$, follows also a homoscedastic process

$$d \ln p_t = (p_0 + w_t)dt + \sigma_p dW_t^p, \quad (3)$$

with $p_0 > 0$ and $\sigma_p > 0$. The mean zero trend growth rate of inflation, w , follows a continuous-time AR(1) process

$$dw = \kappa_w w dt + \sigma_w dW^w, \quad (4)$$

with $\kappa_w < 0$ and $\sigma_w > 0$. I call w the predictable trend component in expected inflation. I account for the negative correlation between z and w through $\sigma_{2z} < 0$. Piazzesi and Schneider (2006) provide macro evidence for a significant negative correlation between consumption growth and inflation. This correlation is also negative in the sample that I use. The realized observable processes z and w are observed in real-time.

Instead of assuming Epstein and Zin (1989) preferences, as done in the long-run risk literature (Bansal and Yaron (2004), Piazzesi and Schneider (2006)), I assume the investor has simple logarithmic utility. This assumption is advantageous for my analysis because it prevents an unwanted amplification of the proposed Knight (1921) uncertainty channel that Epstein and Zin (1989) preferences provide. Of course, the combination of both is

promising in itself, as shown by Drechsler (2009). I leave it for future research to extend the model into that direction. Therefore, the expected life-time utility in period t is $E_t \left[\int_t^\infty e^{-\rho s} \ln c_s ds \right]$, where $\rho > 0$ is the subjective time discount factor.⁸

2.2. Assumptions about Macro Uncertainty

The investor is uncertain about whether (2) and (4) are indeed the true data generating process for z and w . One can say that the investor faces Knight (1921) uncertainty about the long-run risk dynamics. The investor believes that equations (2) and (4) are a good approximation for the unknown data generating process of the long-run risk components. They describe his benchmark model (reference model). The benchmark model accounts for short-run inflation non-neutrality, while also ensuring that inflation has no long-run effect on growth.⁹

The investor observes in each period t a set of potentially correct long-run risk models. Different long-run risk models differ in their conditional expected growth rates and their unconditional variances. After observing z_t and w_t , the investor uses likelihood ratio tests to quantify the accuracy of his trusted benchmark model in comparison to the other models from the

⁸All model implications hold for more general utility functions. It is an advantage of log utility that it supports closed form solutions.

⁹The short-run non-neutrality and long-run neutrality is consistent with Friedman (1968) and Phelps (1968), among others.

set. Due to the stochastic nature of z and w , the investor is exposed to time-varying likelihood ratios and subsequently time-varying trust into the accuracy of his most trustworthy long-run risk model.¹⁰

I define an observed change in the log-likelihood ratio as $d \ln \frac{dQ_t^h}{dQ_t^0}$. An increase in the latter is bad news for the trustworthiness of the benchmark model, because it implies that the observed realization of dz_t and dw_t has probably been generated by the worst-case model Q^h and not by the benchmark model Q^0 . While in general, the investor could focus on every potentially correct model, the Gilboa and Schmeidler (1989) type min-max investor focuses only on the benchmark model and on the worst-case model.

The Gaussian processes z and w imply that the expected instantaneous change of the log-likelihood ratio, under the worst-case measure, is given by

$$E_t^h \left[d \ln \frac{dQ_t^h}{dQ_t^0} \right] = \frac{1}{2} (h_t^r)^2 dt + \frac{1}{2} (h_t^w)^2 dt, \quad (5)$$

where h_t^r is a stochastic perturbation of the dW_t^r shock, while h_t^w is a stochastic perturbation of the dW_t^w shock. Said differently, the agent believes that under the worst-case model, the instantaneous expected growth rate of z_{t+dt}

¹⁰Bansal and Shaliastovich (2010a) study confidence risk in a long-run risk model with Epstein and Zin (1989) preferences and an investor who learns the true data generating process for consumption growth. The authors do not consider Knight (1921) uncertainty.

and w_{t+dt} is $(\kappa_z z_t + \sigma_{1z} h_t^r + \sigma_{2z} h_t^w)dt$ and $(\kappa_w w_t + \sigma_w h_t^w)dt$, respectively. This means that the investor is uncertain whether the predictable long-run risk components z and w are contaminated by empirically difficult to identify stochastic perturbations h^r and h^w . The magnitude and stochastic characteristics of the perturbations h^r and h^w are an equilibrium outcome of the investor's dynamic min-max problem.

Gagliardini et al. (2009) and Gagliardini et al. (2005) propose to constrain the growth rate of the log-likelihood ratio in equation (5) by a joint entropy bound. I slightly deviate from that suggestion in the sense that I use independent entropy bounds for each macroeconomic source of Knight (1921) uncertainty. The separate entropy bounds have two important advantages over a joint entropy bound. First, each source of uncertainty can be treated separately. Second, it enhances analytical tractability and ease of empirical implementation. Mathematically this means that in each period t , the investor quantifies the size of the set of long-run risk model through observing the current magnitude of each entropy bound, i.e.

$$\frac{1}{2}(h_t^r)^2 dt \leq A^r (\eta_t^r)^2 dt, \quad A^r > 0 \quad (6)$$

$$\frac{1}{2}(h_t^w)^2 dt \leq A^w (\eta_t^w)^2 dt, \quad A^w > 0. \quad (7)$$

The last two equations say that the set of models that characterize the data generating process for dW^r and dW^w is time-varying and non-singleton.

It is non-singleton because of $A^r > 0$ and $A^w > 0$. The time-variation in the size of the set of models is characterized by η_t^r and η_t^w . An increase in these processes implies that the set of potentially correct data generating processes for z and w has increased. In such a scenario, the agent would be exposed to a higher degree of uncertainty about the true data generating process for z and w . For simplicity, I assume that both processes follow independent Feller (1951) processes:

$$d\eta_t^r = (a_{\eta^r} + \kappa_{\eta^r}\eta_t^r)dt + \sigma_{\eta^r}\sqrt{\eta_t^r}dW_t^{\eta^r}, \quad (8)$$

$$d\eta_t^w = (a_{\eta^w} + \kappa_{\eta^w}\eta_t^w)dt + \sigma_{\eta^w}\sqrt{\eta_t^w}dW_t^{\eta^w}, \quad (9)$$

with $a_{\eta^r} > 0, a_{\eta^w} > 0, \kappa_{\eta^r} < 0, \kappa_{\eta^w} < 0$ and $\sigma_{\eta^r} > 0, \sigma_{\eta^w} > 0$. The assumption of a Feller (1951) dynamic is smoother compared to the jump-diffusion assumption of Bansal and Shaliastovich (2010a)'s confidence measure. I leave it to future research to allow η to follow a jump-diffusion.

To sum, the investor is uncertain about the transition density of the long-run consumption and long-run inflation risk model. Different models can be characterized by

$$dz_t = \kappa_z z_t dt + \sigma_{1z}(dW_t^{r,h} + h_t^r dt) + \sigma_{2z}(dW_t^{w,h} + h_t^w dt) \quad (10)$$

$$dw_t = \kappa_w w_t dt + \sigma_w(dW_t^{w,h} + h_t^w dt), \quad (11)$$

where $h_t^r \equiv 0$ and $h_t^w \equiv 0$ corresponds to the agent's most trustworthy macro model. The magnitude of potential perturbations that the investor considers reasonable, h_t^r and h_t^w , will be an equilibrium outcome of the agent's min-max problem. In equilibrium they will depend on the observed amount of macro uncertainty η_t^r, η_t^w . Note that h_t^r and h_t^w are endogenously determined stochastic processes who make the transition density of dz_t and dw_t to be non-Gaussian.

2.3. Equilibrium: Endogenous Characterization of the Worst-case Model

The investor has min-max preferences. This implies that he wants to protect himself from the possibility that (2) and (4) are misspecified. The investor wants to find from the set of potentially correct models the single model that implies the lowest expected life-time utility. Such a model is called worst-case model. Formally, the investor solves

$$\min_{Z \in Z(\eta^r, \eta^w)} E^Z \left[\int_t^\infty e^{-\rho(s-t)} \ln c_s ds | \mathcal{F}_t \right] \quad (12)$$

$$s.t. (1), (5), (6), (7), (8), (9), (10), \quad (13)$$

where $Z(\eta^r, \eta^w)$ characterizes the set of potentially correct data generating processes for both long-run risk components z and w .¹¹ The solution to the minimization problem follows Chen and Epstein (2002), Sbuelz and Trojani

¹¹For more details on the mathematical characterization of that set, compare Chen and Epstein (2002).

(2002), Sbuelz and Trojani (2008) and Ulrich (2010). I summarize it in the following proposition.

Proposition 1 *Given the observed set of potentially correct long-run risk models, i.e. $Z(\eta_t^r, \eta_t^w)$, the solution to (12) tells the investor that the long-run risk model with the lowest expected life-time utility has an instantaneous drift of $(\kappa_z z_t + \sigma_{1z} h_t^r + \sigma_{2z} h_t^w) dt < \kappa_z z_t dt$, where h_t^r and h_t^w are endogenous and deterministic functions of the amount of observed macro uncertainty, i.e.*

$$h^r(t) = m^r \eta^r(t), m^r := -\sqrt{2A^r} \in \mathcal{R}^- \quad (14)$$

$$h^w(t) = m^w \eta^w(t), m^w := \sqrt{2A^w} \in \mathcal{R}^+. \quad (15)$$

The proof of Proposition 1 is in the appendix. The proposition implies that the worst-case long-run consumption risk model has a lower expected instantaneous growth rate compared to the forecast from the reference model in (2). The instantaneous difference is $(\sigma_{1z} m^r \eta_t^r + \sigma_{2z} m^w \eta_t^w) dt < 0, \forall t > 0$. The observed size of the set of potentially correct models, i.e. η_t^r and η_t^w , is stochastic. The density of dz is therefore highly non-Gaussian under the worst-case long-run risk model.

2.4. Equilibrium: Marginal Rate of Substitution

Consistent with Hansen and Sargent (2008) and Chen and Epstein (2002), the pricing of assets is done under the most trustworthy reference model Q^0 . The key difference to Rational Expectations is that the stochastic discount factor will depend on the amount of model mistrust. This is very intuitive because the investor mistrusts his reference model and that mistrust has to affect his discount factor.

Alternatively, one can re-write the solution to the min-max problem in equation (12) as

$$U^h(c_0; \eta_0^r, \eta_0^w) := E_0 \left[\int_0^\infty e^{-\rho t} a_t \ln c_t dt \right] \quad (16)$$

$$\frac{da_t}{a_t} = h_t \cdot dW_t^z \quad (17)$$

where a_t is the Radon-Nikodym derivative between the worst-case and the reference long-run risk model, $h_t = (h_t^r \ h_t^w)'$ is derived in Proposition 1, and $dW_t^z = (dW_t^r \ dW_t^w)'$ are shocks to macro uncertainty. In that specification, a_t appears to be a time-varying and endogenously constrained preference shock (Hansen and Sargent (2008)).

Equation (17) and Proposition 1 imply that an unpredictable reduction in dW^r , increases the investor's doubts about whether his most trustworthy

model (2) is indeed the correct data generating process. This is very intuitive because since the model for z is not known, a lower than anticipated growth rate for z_t , i.e. $z_{t+dt} < E_t[dz_t]$, increases the statistical evidence that the worst-case model (10) generated the data, i.e. $E_t^h[dz_t] < E_t[dz_t]$. Lower than expected growth hurts a min-max preference investor twice. First, consumption growth is lower than expected and second, the investor is in a state of higher model uncertainty.

On the other hand, equation (17) and Proposition 1 show that an unpredictable increase in trend inflation, i.e. $dW^w > 0$, increases the investor's fear that the most trustworthy inflation model in equation (4) is misspecified. This mechanism is intuitive because the worst-case inflation model predicts higher growth rates for trend inflation, compared to the reference model in equation (4). This mechanism was first analyzed by Ulrich (2010), who uses that insight to explain why the nominal term premium is upward sloping. In contrast to that study, I focus in this paper on bond premiums, bond volatility, options and the relative importance of GDP vs. inflation ambiguity.

Equation (16) shows that the intertemporal marginal rate of substitution (MRS) accounts for consumption risk and for model misspecification doubts about the consumption model. I define the MRS as m . Besides its tractable shape, I emphasize that m is the endogenous equilibrium outcome of the

dynamic min-max optimization problem in Proposition 1, i.e.

$$m_{t,t+\Delta} = e^{-\rho\Delta} \left(\frac{c_{t+\Delta}}{c_t} \right)^{-1} \frac{a_{t+\Delta}}{a_t}. \quad (18)$$

The equilibrium MRS of a min-max agent accounts not only for the standard consumption risk kernel, but also for a Knightian uncertainty kernel (Chen and Epstein (2002), Ulrich (2010)). The uncertainty kernel quantifies the statistical confidence that the investor has in his most trustworthy long-run risk model, i.e. (2), (4). The uncertainty kernel is driven by shocks to W^r and W^w . Uncertainty about the long-run risk model is therefore a natural framework for endogenizing that shocks to the long-run risk component are priced. Interestingly, this result does not require Epstein and Zin (1989) preferences.

The evolution of the MRS reveals the equilibrium real interest rate and the equilibrium market prices for risk and uncertainty:

$$-\frac{dm_t}{m_t} = \frac{dc_t}{c_t} - \frac{da_t}{a_t}. \quad (19)$$

According to the consumption dynamic in equation (1), the market price of consumption risk is constant and positive, i.e. $\sigma_c > 0$. The endogenous dynamic in (17) reveals that the market price for uncertainty about the consumption model has two components, i.e. $-h_t^r \in \mathcal{R}^+, \forall t \geq 0$ and

$-h_t^w \in \mathcal{R}-, \forall t \geq 0$. The first is the market price for uncertainty about whether dW_t^r is misspecified, while the latter is the market price for uncertainty about whether dW_t^w is misspecified. The different signs indicate that uncertainty about real vs. nominal shocks has different economic implications. I analyze the different implications in the bond section. The real interest rate coincides with $r_t = \rho + c_0 - \frac{1}{2}\sigma_c^2 + z_t$.

Having derived m allows the determination of the nominal SDF $m^{\$}$, i.e.

$$m_{t,t+\Delta}^{\$} = m_{t,t+\Delta} \frac{p_t}{p_{t+\Delta}}. \quad (20)$$

The equilibrium nominal short rate coincides with

$$R_t := -E_t \left[\frac{dm_{0,t}^{\$}}{m_{0,t}^{\$}} \right] = r_t + w_0 - \frac{1}{2}\sigma_p^2 + w_t.$$

The real and the nominal short-rate do not depend on macro uncertainty, whereas all interest rates with a non-zero maturity depend on macro uncertainty. This is intuitive because non-zero maturity interest rates are risk and uncertainty adjusted expectations of future short rates. I use the model insight that interest rate data reveals the characteristics of macro uncertainty premiums, which would otherwise be hidden in the investor's marginal utility.

2.5. Equilibrium: Bond Market

The equilibrium bond market reveals valuable information about the investor's marginal utility. Let $B_t(\tau)$ be the price at time t of a τ maturity inflation-indexed bond. Based on a standard Euler equation, its price coincides with the conditional expected growth rate of marginal utility between t and $t + \tau$. The separate entropy bounds in equation (6) and (6) make the bond price fall into the class of exponentially affine models (Duffie et al. (2000)):

$$B_t(\tau) = e^{A^r(\tau) + B^r(\tau)S_t}, \quad S \equiv (z \ h^w \ h^r)'. \quad (21)$$

The last equation reveals that the price of a real bond is exponentially affine in expected consumption growth and in the two market prices for uncertainty about the correctness of the most trustworthy consumption model. Both uncertainty premiums affect the price of real bonds because they affect the agent's trust in the inference of future consumption growth that comes out of the reference model in equation (2).

The continuously compounded real interest rate is affine in the z_t and h_t , i.e.

$$y_t^r(\tau) = a^r(\tau) + b^r(\tau)S_t, \quad (22)$$

with $a^r(\tau) := -A^r(\tau)/\tau$ and $b^r(\tau) := -B^r(\tau)/\tau$. The loadings are deterministic functions of the parameters of the economy. The appendix contains details on the derivation.

I denote the price at time t of a τ maturity nominal Treasury bond as $N_t(\tau)$. Its equilibrium price is exponentially affine in S_t and in the predictable component of inflation, i.e. w_t :

$$N_t(\tau) = e^{A^n(\tau) + B^n(\tau)X_t}, \quad X \equiv (w S)', \quad (23)$$

with $a^n(\tau) := -A^n(\tau)/\tau$ and $b^n(\tau) := -B^n(\tau)/\tau$.

Nominal interest rates are driven by variations in z and w , as well as the corresponding uncertainty premiums h_t^r and h_t^w . Mathematically, this means

$$y_t^n(\tau) = a^n(\tau) + b^n(\tau)X_t, \quad (24)$$

with $a^n(\tau) := -A^n(\tau)/\tau$ and $b^n(\tau) := -B^n(\tau)/\tau$. The appendix specifies the factor loadings as functions of the underlying economy.

The slope of the nominal yield curve, measured as the 40-quarter nominal

yield minus the nominal short-rate, is an affine function in w , z , and h_t :

$$y_t^n(40) - R_t = (b_w^n(\tau) - 1)w_t + (b_z^n(40) - 1)z_t + b_{h^r}^n(40)h_t^r + b_{h^w}^n h_t^w. \quad (25)$$

The bond loadings indicate how macro risk and uncertainty about the trustworthiness of the risk model affects Treasury yields of all maturities.

While the yield curve is affected by macro risk and macro ambiguity, the model implied instantaneous bond premium depends only on macro uncertainty:

$$\begin{aligned} E_t^h \left[\frac{dN_t(\tau)}{N_t(\tau)} - R_t dt \right] &= (\sigma_{2z} B_z^n(\tau) + \sigma_w B_w^n(\tau))(-h_w(t))dt + \\ &+ \sigma_{1z} B_z^n(\tau)(-h^r(t))dt. \end{aligned} \quad (26)$$

States of the world in which the agent has more doubts about the accuracy of the long-run inflation model are states where h_t^w increases. This coincides with an increase in the expected excess return of a nominal bond. On the other hand, an expected increase in uncertainty about the data generating process of dW^r , lowers h^r , which reduces the bond premium across all maturities.

The type of uncertainty matters for the bond premium. While bonds hedge dW^r uncertainty, they do not hedge uncertainty about dW^w . This im-

plies a positive bond premium for inflation uncertainty and a negative bond premium for consumption uncertainty. The equilibrium bond premium can have different signs, depending on which type of macro uncertainty dominates.

The volatility of interest rate changes is time-varying as well. If the dynamics of the underlying macro economy become more uncertain, real and nominal bond yields exhibit higher heteroscedasticity. My model shows that this holds even if the underlying inflation and consumption dynamic has not changed. The time-varying amount of trust in the workhorse model is the equilibrium channel that my model adds to the literature. The quadratic variation of changes in yields depends on the heteroscedastic uncertainty premiums. Mathematically this means

$$\langle dy_t^n(\tau), dy_t^n(\tau) \rangle = (b_{hr}^n(\tau)m^r\sigma_{\eta^r})^2 \eta_t^r dt + (b_{hw}^n(\tau)m^w\sigma_{\eta^w})^2 \eta_t^w dt \quad (27)$$

$$\langle dy_t^r(\tau), dy_t^r(\tau) \rangle = (b_{hr}^r(\tau)m^r\sigma_{\eta^r})^2 \eta_t^r dt + (b_{hw}^r(\tau)m^w\sigma_{\eta^w})^2 \eta_t^w dt \quad (28)$$

where b_{hr} and b_{hw} are the corresponding yield loadings for h_t^r and h_t^w .

2.6. Equilibrium: Option Market

Option markets reveal the higher moment properties of marginal utility. It is natural to anticipate that uncertainty about the reliability of the macro model manifests itself into the higher moments of marginal utility. In the

model, bond options contain in equilibrium a time-varying implied volatility which itself consists of η_t^r and η_t^w . Changes in option implied volatilities arise because the investor's perception about the trustworthiness of the reference model in (2) and (4) has changed. This implies that variations in macro uncertainty lead to endogenous fluctuations in the option implied volatilities.

The precise type of the bond option is secondary. I consider an interest rate option that is traded on the CBOE. Let C be a floor on an interest rate $y^n(\tau)$. Being long a floor entitles the owner of the option to receive at maturity T the maximum of $(y_T(\tau) - K, 0)$, where K is a fixed interest rate level. The value of this option depends on the maturity of the contract, strike, current value of the yield of interest and the current nominal short rate. I fix the notional of the contract to \$100. Its equilibrium price is

$$C(t, T, K, y_t^n(\tau), R_t) := \$100 E_t (m_{t,T}^{\$}(y_T^n(\tau) - K)^+). \quad (29)$$

The same model with Rational Expectations implies that the price of the option follows a Black (1976) type model. The implied volatility would be constant and the model would fail to reproduce the skewed option smile that is observed in the data. Accounting for misspecification doubts about the most trustworthy macro model endogenizes option prices that follow a Heston (1993) type dynamic. Variations in the trust of whether the benchmark model is a good description of reality leads to variations in option implied

volatilities. The model can account for smiles and skews in the option market, because the size of the set of models is time-varying.¹² The casual interpretation of implied volatility as a fear (uncertainty) index is adequate in this model because implied volatility is an endogenous reaction to uncertainty about the dynamics of the macro economy.

In the empirical section, I determine the price of a call option via Monte Carlo simulation. I use an Euler-Marujama discretization scheme, and simulate the system on a daily interval. I plug the resulting call price into Black (1976) formula to recover the option implied volatility.

3. Empirical Part

The model provides intuition for how uncertainty about the reliability of the underlying macro model feeds into variations in marginal utility and the bond market. It is not the goal of the paper to build the most advanced term structure model that minimizes pricing errors. Instead, the goal is to analyze how macro uncertainty feeds into bond premiums and bond variances. A

¹²Recent equilibrium models have exclusively focused on equity options. Liu et al. (2005) and Drechsler (2009) show that ambiguity about rare events can explain the skewed volatility smile in equity options. Buraschi et al. (2009), David and Veronesi (2009), Buraschi and Jiltsov (2006), David and Veronesi (2002) show that learning about the fundamental processes in the economy helps to explain why dispersion in forecasts explain equity option prices. Drechsler and Yaron (2010), Bollerslev et al. (2009b), Eraker and Shaliastovich (2008), Bollerslev et al. (2009a), Shaliastovich (2009) show that stochastic volatility in consumption growth together with Epstein-Zin preferences can explain equity option prices.

more complex model provides a closer fit to data, but makes it less intuitive to interpret the equilibrium effect of model misspecification doubts.¹³

I use maximum-likelihood to estimate the model with macro, bond yield, and bond variance data. The estimation results provide a pedagogical tool to visualize the effect that model misspecification doubts have on expected bond returns, as well as the smile and skew of option implied volatilities.

3.1. Data and Estimation

The data is from 1972 to 2009. The data frequency is quarterly. I use the following data to match the exogenous processes of the economy. First, $d \ln c$ is matched with realized GDP growth. I use GDP data instead of consumption data, because there is no forecast and model disagreement data for consumption growth available. The intuition for the model are not affected by that approximation.¹⁴ Second, $d \ln p$ is matched with realized inflation. Third, z is matched with the demeaned median forecast of GDP growth over the next quarter. The forecast is taken from the Survey of Professional Forecasters (SPF). Analogously w is matched with the demeaned median SPF forecast for inflation over the next quarter. Fourth, η^r (η^w) coincides with the cross-sectional standard deviation of one-quarter ahead SPF GDP (infla-

¹³Recent examples for successful reduced-form term structure models are Ang et al. (2010), Chernov and Mueller (2008), Chun (2007), Joslin et al. (2009), ?, among others.

¹⁴Moreover, it is a representative agent model, where the investor consumes the entire output stream. It is standard that these models do not distinguish between output and consumption.

tion) forecasts.

Patton and Timmermann (2010) find evidence that the cross-sectional dispersion in macro forecasts is a reliable proxy for the amount of model disagreement.¹⁵ I remove seasonality in η_t^w and η_t^z by using a 4-quarter moving average to construct the non-seasonal t -measure for η^w and η^r .¹⁶ The appendix contains a more detailed description of the macro data.

I match the model output with the following financial data. First, ten panels of continuously compounded nominal Treasury bond yields of maturity one year to ten years. Second, six panels of continuously compounded real Treasury bond yields of maturity five years to ten years. Real bond yields coincide with yields of Treasury Inflation Protected Securities (TIPS). Third, ten panels of variances of continuously compounded nominal Treasury yields with maturity of one year to ten years. The quarterly variance within a quarter is estimated as the sum of quadratic daily yield changes within the corresponding quarter. The appendix contains details on the exact construction of the financial data.

¹⁵Anderson et al. (2009) and Ulrich (2010) use dispersion data to quantify the amount of model ambiguity. Research on differences in belief rely also on dispersion data to identify belief disagreement. Compare Buraschi et al. (2009), Buraschi and Jiltsov (2006), David and Veronesi (2002), David and Veronesi (2009), among others.

¹⁶Results are robust with regard to that smoothing.

I assume that all measurement errors of the financial variables are orthogonal to each other (Chen and Scott (1993), Duarte (2000)). A standard maximum likelihood approach is applied to estimate the parameters of the model. A clear advantage of the proposed model mechanism is that GDP and inflation uncertainties are not treated as latent variables, but instead are taken to be observable. Filtering of latent processes, as would be required in a stochastic volatility set-up, is not required.

A first look at the macro data reveals that z and w are unbiased predictors of realized GDP growth and realized inflation, respectively. The former explains 9% of the variance of realized GDP growth, while the latter explains 65% of the variance of realized inflation.

A first look at the bond data reveals interesting relations between the macro and the bond market. I denote the principal components of the panel of nominal yields as PC. PC1 stands for the first principal component, PC2 stands for the second, and so on. First, variations in $X_t = (w_t z_t \eta_t^r \eta_t^w)$ explain 63% of PC1 variations, with w_t and η_t^w having t-stats bigger than 2.

Second, variations in X explain variations in PC2, with w_t and η_t^r having t-stats bigger than 2. Third, variations in X explain 19% of PC3, with w_t and η_t^w having t-stats bigger than 2. Fourth, variations in X explain 4% of PC4

variations, while z_t has a t-stat of bigger than 2. Fifth, 12% of variations in PC5 are explained by variations in η^w . Sixth, 23% of variations in the slope (40-quarter yield minus federal funds rate) are explained by w_t and η_t^r . The negative loading on these factors is consistent with the model implied sign in equation (25).

The data evidence suggests that uncertainty about the inflation model explains variations in PC1, PC3, and PC5. It further implies that uncertainty about trend GDP growth explains variations in PC2. Cochrane and Piazzesi (2005) find that a single factor of forward rates predicts bond returns. This factor is mostly unrelated to the first three principal components of nominal yields. Duffee (2010) finds that the fifth principal component is a hidden factor in the yield curve which has substantial predictive power for bond returns. My first look at the data reveals that cross-sectional dispersion in inflation forecasts, which is part of the bond premium in my model, explains 12% of variations in the fifth principal component.

A first look at the volatility of yield changes reveals the following. First, one factor explains 94% of fluctuations in the variance of bond yields. Second, regressing this factor on $(\eta^w)^2$ and $(\eta^r)^2$ reveals that both factors are highly significant and they jointly explain 20% of its variations.

3.2. Empirical Findings

Observable uncertainty about misspecified long-run risk dynamics. Figure 1 plots the four observable macro states of the economy together with NBER recession dates. The upper panel reveals the negative correlation between both long-run risk components. Macro risk z and w have moved in opposite directions, especially during the last six recessions. The unconditional correlation in the data between z and w has been -0.3231 . My model captures this through a negative estimate of σ_{2z} . The negative correlation of trend growth and trend inflation is consistent with Piazzesi and Schneider (2006) who document a similar finding.

The lower panel of Figure 1 plots the annualized cross-sectional standard deviations of GDP growth and inflation forecasts. This dispersion characterizes time-variation in the set of models, (6) and (7). I document that the exogenous set of long-run risk models is larger for the GDP component, whereas inflation uncertainty is more persistent. It is only during the past financial crisis that the set of long-run inflation models has been as large as the set of long-run GDP models. Consistent with intuition, uncertainty spikes during recessions and falls during expansions. The Great Moderation is also clearly visible in macro uncertainty.¹⁷ GDP and inflation uncertainty have been falling since 1985. Macro uncertainty doubled since early 2005,

¹⁷Stock and Watson (2002) document falling volatilities across several macro variables since the mid 1980s.

which is mainly related to the strong increase during the recent financial crisis.

Effect of z, w and η^r, η^w on bond prices. The model focuses only on the predictable (and observable) components of GDP growth and inflation, as well as on their measurable amount of model misspecification doubts. The goal is to derive intuition on how these macro concepts affect marginal utility and the bond market. Table 2 summarizes the fit to the financial measurement equations. The estimation procedure matches GDP growth, inflation, and the four macro states perfectly. The model fits the level and slope of the nominal yield curve. In a simple log-utility set-up, this relies on accounting for inflation uncertainty, as explained in Ulrich (2010). Long-run risk models, such as Piazzesi and Schneider (2006), Piazzesi and Schneider (2010) and Bansal and Shaliastovich (2009) match the nominal term spread as well. Their economic mechanism is an inflation risk premium that gets amplified through Epstein and Zin (1989) preferences.

The fit to the real yield curve and to the yield volatility has a higher measurement error. The result for the real yield curve is not surprising because D'Amico et al. (2008) find evidence for a liquidity factor in TIPS which is not present in my model. Moreover, equilibrium yield curve models in general

have difficulties explaining the positive slope in TIPS.¹⁸ The higher measurement error for the yield volatility has two reasons. First, realized variance that is constructed from daily data is very noisy. The confidence bounds for the mean term structure of yield volatility is large. The model estimates fall within that bound. Second, for simplicity and in order to focus the attention to model misspecification doubts about the long-run risk components, I have abstracted from stochastic volatility in consumption growth and inflation. Adding stochastic volatility as in Drechsler and Yaron (2010) or Bekaert et al. (2009) would reduce the measurement error but also destroy the simplicity of the model intuition. I therefore regard the measurement error in bond volatility as a latent volatility factor that is orthogonal to long-run macro ambiguity and that could be explained by stochastic volatility in consumption growth and inflation.

While Figure 1 documents that the set of long-run GDP models is, relatively speaking, larger than the set of long-run inflation models, I also find that inflation ambiguity dominates the investor's marginal utility. Said differently, a bond investor is really concerned about slight misspecifications of dW^w .

¹⁸The real yield curve in Piazzesi and Schneider (2006), Piazzesi and Schneider (2010) and Bansal and Shaliastovich (2009) is downward sloping as well.

What are reasons for this finding? The relative importance of w over z is one explanation for why bond investors are more concerned about using the wrong model for w , compared to using the wrong model for z . Panel A of Table 3 shows that w explains most of the variations in U.S. Treasury yields. For short maturity bonds (4-quarter), w explains 74% of variations. Most of the remaining variations are captured by z . The dominance of w is even stronger for long-term bonds (40-quarter), where w accounts for nearly all variations. This is consistent with Ang et al. (2008) and Gürkaynak et al. (2005) who conclude that expected inflation is one of the most important drivers of long-term yields.

According to (24), macro risk (w, z) and the uncertainty premiums for model misspecification doubts affects U.S. Treasury yields. The variance decomposition in the top panel of Table 3 reveals that the uncertainty premiums appear to be unspanned by the yield curve. These endogenous equilibrium premiums affect expected bond returns and option volatilities, (26) and (29), but they virtually do not help to explain variations in bond yields. The equilibrium mechanism of having model misspecification doubts about the data generating process of w and z endogenizes two unspanned macro factors that drive bond returns and bond volatilities. This provides an equilibrium explanation to the challenging empirical findings of Collin-Dufresne and Goldstein (2002), Joslin et al. (2009), Cochrane and Piazzesi (2005), Duffee (2010), and Ludvigson and Ng (2009) who find that factors that are unspanned by the

yield curve drive bond returns and bond volatilities. To the best of my knowledge, I am not aware of any other equilibrium model that endogenizes unspanned premiums and unspanned volatilities.

Uncertainty premiums appear unspanned in a variance decomposition, but they do affect bond prices, but indeed only marginally. Figure 2 summarizes the impulse response of short- and long-term interest rates to a one percent shock to η^r and η^w . They reveal that η^r affects bond prices very differently than η^w .

An increase in η^r means the investor has more empirical evidence that his model for z , i.e. (2), is potentially misspecified. An ambiguity averse agent worries that future growth might be lower than what the benchmark model ((2)) predicts. The endogenous response is to become more bearish about future growth. Bonds are recession hedges and more attractive in such uncertain states, because their payout is fixed. The impulse response reveals that a one percent increase in η^r leaves R unchanged and lowers $y^n(40)$ by 0.003%. This means long-term bond prices go up in value, because of flight-to-safety. This economic channel is qualitatively very intuitive and novel to the literature. The small quantitative effect is a direct consequence of the unspanned nature of η^r

An increase in η^w affects the economy very differently. In fact, the investor has more fear that his most trustworthy forecast for w ((4)) is too low. This means he has more empirical evidence to fear that future inflation will be drawn from a distribution with a higher mean and a higher variance ((11)). If this was indeed true, the future real payout of a U.S. Treasury bond will be lower in a state of higher marginal utility. The attractiveness of a Treasury bond falls which translates into falling bond prices. The impulse response of Figure 2 confirms this intuition. A one percent increase in η^w leaves R unchanged and increases $y^n(40)$ by 0.009%. The increase in yields is a response to falling bond prices. It is the virtually unspanned feature of η^w that makes the quantitative response small, but economically meaningful.

Effect of z, w and η^r, η^w on bond premiums. The unspanned uncertainty premiums h^r, h^w drive variations in bond premiums. These premiums depend linearly on the observed amount of macro uncertainty, η^r and η^w , as shown in Proposition 1. Figure 3 summarizes that both unspanned uncertainty premiums affect bond returns qualitatively very differently.

A one percent increase in η^w increases the equilibrium bond premium for a long-term bond (40-quarter) by 1.5%. This 1.5 multiple indicates that misspecification doubts about dW^w has a quantitatively important effect on equilibrium bond premiums. The reason for the increase is that the investor

has a stronger concern about the prospect that the future real payout will be lower than forecasted under the reference model. This lower payout will happen in a period of higher marginal utility.

On the other hand, a one percent increase in η^r reduces the bond premium of a long-term bond by 0.025%. The economic reason for the negative bond premium is that bonds hedge GDP uncertainty. The quantitative impact on bond premiums is smaller, compared to an increase in η^w . Together with the lower panel of Figure 1 this shows that investors are really concerned about working with a misspecified long-run inflation model. This is consistent with Ang et al. (2008), Gürkaynak et al. (2005) and Ulrich (2010) who document the importance of inflation for bond markets.

Analyzing Figure 3 and Figure 2 at once, shows that an increase in η^w increases the slope of the yield curve, as well as the expected bond premium. This is consistent with the well known empirical phenomenon that a steepening of the yield curve coincides with higher expected bond returns (Campbell and Shiller (1991), Cochrane and Piazzesi (2005), among others).

Term structure of unspanned bond premiums. Figure 4 compares realized annual bond excess returns with the model's prediction. The bond data from the sample allows the exact measurement of annual holding period

returns for all bond maturities. I construct excess returns by subtracting the federal funds rate.¹⁹ As the model counterpart, I use the integrated annual expected excess return.²⁰

In the data, the term structure of annual excess returns is upward sloping. It is roughly zero for a short-term bond and 2.2% for a long-term bond. The term structure of annual excess returns is volatile in the data. Three standard error bounds put the population mean of long-term excess returns between -0.86% and 5.21% .

The unspanned macro uncertainty premiums in the model explain the upward sloping bond premiums. In terms of magnitude, it lies in the upper part of the data implied three standard error confidence bounds. How does this result compare to competing models? Bond models with Epstein and

¹⁹I use the federal funds rate as the risk-free rate and not the four-quarter yield. This is the closest counterpart to the model implied expected excess return.

²⁰I find no difference between the integrated and the instantaneous expected excess return. The unconditional correlation between both is higher than 98%. The integrated annual expected excess return is

$$\begin{aligned}
& \int_t^{t+4} d \left(E_t^h \left[\frac{dN_t(\tau)}{N_t(\tau)} - R_t dt \right] \right) \\
&= (\sigma_{2z} B_z^n(\tau) + \sigma_w B_w^n(\tau)) \left(\left(\frac{m_w a_{\eta^w}}{\kappa_{\eta^w}^2} - \frac{h_t^w}{\kappa_{\eta^w}} \right) (e^{\kappa_{\eta^w} \cdot 4} - 1) + 4 \frac{m_w a_{\eta^w}}{\kappa_{\eta^w}} \right) \\
&+ \sigma_{1z} B_z^n(\tau) \left(4 \frac{m_r a_{\eta^r}}{\kappa_{\eta^r}} + \left(\frac{m_r a_{\eta^r}}{\kappa_{\eta^r}^2} - \frac{h_t^r}{\kappa_{\eta^r}} \right) (e^{\kappa_{\eta^r} \cdot 4} - 1) \right)
\end{aligned}$$

Zin (1989) preferences as in Piazzesi and Schneider (2010) and Bansal and Shaliastovich (2009) or Campbell and Cochrane (1999) type models as in Wachter (2006), Bekaert et al. (2009) and Buraschi and Jiltsov (2007) do not analyze the impact on bond excess returns. I leave it to future research to compare these models on that dimension.

Panel B of Table 3 presents a variance decomposition of annual bond excess returns. The unspanned nature of bond premiums is evident. Whereas variations in long-run inflation and GDP risk (w, z) dominate variations in bond yields, they have a zero impact on variations in bond premiums. Bond premiums are exclusively driven by variations in uncertainty about the quality of the benchmark long-run risk model. More than 90% of variations in the bond premium are driven by η^w . Only 10% are due to variations in η^r . This confirms that the investor really dislikes the potential threat of working with the wrong inflation model. It also confirms standard intuition that periods of increased doubt about the underlying model are periods where asset premiums (prices) increase (fall).

Option markets. Collin-Dufresne and Goldstein (2002) and Heidari and Wu (2003) argue that option implied bond volatilities are driven by a factor that does not help to explain variations in the yield curve. My model argues that the unspanned factors η^r and η^w should help to explain part of

option implied volatilities. Times of increased uncertainty are times where bond volatilities go up which translates into higher option implied volatilities. The equilibrium channel in my model has important economic implications. First, if the agent knew the data generating process of z and w , the equilibrium Black (1976) implied bond option volatility would be constant.

Second, concerns about whether (2) and (4) are the true data generating processes, make the equilibrium bond option volatility to be of Heston (1993) type. Heston (1993) derives a reduced-form option pricing model that supports stochastic option implied volatilities. In that model, volatility of the underlying process is stochastic. Reduced-form option pricing models confirm that option implied volatilities are time-varying (Broadie et al., Chernov and Ghysels (2000), Pan (2002), Trolle and Schwartz (2000), Eraker (2004)), which falsifies the constant volatility model of Black (1976). Collin-Dufresne and Goldstein (2002) added an additional puzzle because they found that option implied volatilities do not depend on the underlying. So far, equilibrium models have not been able to explain why option implied bond volatilities are unspanned by the yield curve. My equilibrium model argues that uncertainty about the correctness of the reference model in equation (2) and (10) induces stochastic volatility that appears to be unspanned by the yield curve.

How could I test my model on that dimension? The Chicago Board

Option's Exchange's (CBOE) VIX index is a summary statistic for several short-term *S&P500* options. It is usually referred to as the 'fear index'. In addition, Bloomberg offers historical data on the implied volatility from options on the 10-year T-note futures contract.²¹ A correctly specified equilibrium model should be able to explain part of the variations in TIV and maybe even in the VIX.

Implied volatility in my model depends on the unspanned factors η^r and η^w . I combine their information by constructing

$$\theta_t := \sqrt{(\eta_t^r)^2 + (\eta_t^w)^2}, \quad (30)$$

because implied volatility is only one process which is an aggregate of both contributors.

I take quarterly data on the TIV for 1994:Q1 to 2009:Q2.²² I regress it on θ_t and find that the model implied measure for macro uncertainty helps a great deal to explain variations in the TIV

$$TIV_t = 0.0084 + 8.1507\theta_t + \epsilon_t, \quad R^2 = 45.63\%. \quad (31)$$

(2.26) (7.06)

²¹In Bloomberg this corresponds to the TY1 series.

²²The data is not available before 1993:Q3.

Roughly half of variations in bond options are related to changes in uncertainty about the data generating process that drives the trend component in GDP growth and in inflation. This confirms the prediction of my model. The t-stats in parentheses show that an increase in macro uncertainty leads to a significant increase in option implied volatilities. My simple equilibrium model helps to understand variations in the option markets.

I repeat the exercise for the VIX for 1994:Q1 to 2009:Q2. The regression output reveals that unspanned macro uncertainty explains roughly 50% of variations in the VIX

$$\begin{aligned}
 VIX_t = & -0.0123 & + & 37.0 \theta_t + \epsilon_t, & R^2 = 45.75\%. \\
 & (-0.07295) & & (7.11) &
 \end{aligned} \tag{32}$$

The high explanatory power of macro uncertainty for equity options and bond options is consistent with the notion that option volatilities depend on factors that do not help to explain variations in the underlying. It is also consistent with the popular interpretation that an increase in VIX or TIV happens because investor's fear has gone up. My model argues that in states where the investor becomes more uncertain about whether his most trustworthy macro model might be misspecified are states where VIX and TIV increase. This equilibrium channel is new to the literature and adds an alternative perspective to existing equilibrium models (Drechsler (2009), Drechsler and Yaron (2010), David and Veronesi (2002), Buraschi and Jiltsov

(2006), David and Veronesi (2009), Benzoni et al. (2011), Tauchen (2005)).

Option smile Predictions. Figure 8 confirms that my simple model can indeed generate a Black (1976) implied volatility smile and smirk. The model argues that options on a short-term bond have a higher and more skewed implied volatility than the same option on a long-term bond. In order to construct the picture, I have priced calls on Treasury yields. The calls have a maturity of three months and are priced as $\$100E_t [m_{t+1}^{\$} \max(y_{t+1}^n(\tau) - K, 0)]$. The maturity of the bond, τ , varies from 8 quarters to 20 quarters. I determine the model implied equilibrium price of the call for different strikes. This means that the discount factor $m^{\$}$ is the nominal equilibrium SDF from equation (20).

4. How much Uncertainty is Necessary

If the worst-case ((10), (11)) and the reference model ((2), (4)) are in a statistical sense far apart from each other, it becomes easy for an econometrician to tell which model generated the data. The amount of model uncertainty would be low in such a scenario. I therefore, determine the detection error probability (DEP), evaluated at the ML estimates. This probability denotes the likelihood, that a likelihood ratio test favors one model, although the data has been generated by the other model. Ulrich (2010) explains in detail how to derive DEPs in macro-finance models, and how they relate to

Hansen and Sargent (2008).

The time-varying log-likelihood ratio between the worst-case model and the benchmark model is

$$\ln \left(\frac{dQ_T^h}{dQ_T^0} \right) = -\frac{1}{2} \int_0^T ((m^r)^2 (\eta_t^r)^2 + (m^w)^2 (\eta_t^w)^2) dt + \int_0^T (m^r \eta_t^r dW_t^r + m^w \eta_t^w dW_t^w). \quad (33)$$

The detection error probability depends on the market price of uncertainty and the realization of shocks to both long-run risk components. The appendix provides details on the derivation of the DEP.

The DEP, evaluated at the ML estimates is 23.5%. This says that after seeing the data, if the investor was to choose whether the data has been generated by the worst-case ((10), (11)) or the reference model ((2), (4)), the likelihood ratio test would fool the investor in 23.5% of all cases. Hansen and Sargent (2008) argue that difficult to distinguish models should have a DEP of at least 10%.

My model mechanism reveals that being exposed to uncertainty about the data generating process of the long-run risk components helps to explain why bond premiums and bond option implied volatilities are time-varying and unspanned. In a bigger context this reveals that accounting for model

misspecification doubts helps to explain higher moments of the investor's marginal utility.

5. Conclusion

We know from Campbell and Cochrane (1999) that stochastic risk aversion can explain a counter cyclical equity premium, while the conditional volatility of consumption is constant. I extend this reasoning and show that stochastic uncertainty aversion (Knight (1921)) is an alternative channel for the bond market. In the model, the equilibrium bond premium, volatility, and option implied volatility are counter cyclical, while conditional volatility of consumption and inflation is constant. The reason are time-varying misspecification doubts about the statistical reliability of the underlying macro model. These misspecification doubts induce endogenous uncertainty premiums.

The analysis concludes that the yield curve does not span the uncertainty premiums. At the same, these premiums are of first-order importance for understanding bond returns and option markets in this model. The model provides a general equilibrium rationale for the empirical findings that unspanned factors drive bond premiums, bond volatilities, and option implied volatilities (Collin-Dufresne and Goldstein (2002), Cochrane and Piazzesi (2005), Ludvigson and Ng (2009)). I further document that the model accounts for 50% of the variations in the VIX and the TIV.

My equilibrium model shows that accounting for Knight (1921) uncertainty about the data generating process of the predictable components in GDP growth and inflation is helpful for generating highly non-Gaussian return distributions, even if the fundamentals are smooth processes. My endogenous equilibrium result is promising because reduced-form option pricing models find strong skewness in option contracts.²³ The proposed equilibrium channel and bond market focus adds to existing models, who exclusively focus on equity options (David and Veronesi (2002), Liu et al. (2005), Benzoni et al. (2011), Drechsler and Yaron (2010), Drechsler (2009) and Shaliastovich (2009), David and Veronesi (2011), among others).

I conclude that the introduced small deviation from an otherwise conditionally homoscedastic consumption based asset pricing model has important and realistic implications for bond and bond option markets. For future research it is promising to study stochastic risk aversion, stochastic uncertainty aversion and Epstein and Zin (1989) preferences in a unified model.

²³Compare Chernov and Ghysels (2000), Pan (2002), Broadie et al., among others.

Appendices

A. Data

Macro data:

The Survey of Professional Forecasters (SPF) does not publish forecasts on consumption growth. I use forecasts and dispersion on GDP growth instead. Real GDP growth, GDP implicit price deflator, federal funds rate are from the St. Louis Fed database (FRED). The quarterly forecast on GDP growth and inflation coincide with the corresponding median forecast from the SPF. For each quarter, I determine the amount of ambiguity, η^2 , as the cross-sectional variance of SPF's inflation and GDP growth forecasts. To remove seasonality I use a 4-quarter moving average. All data is from first quarter 1972 to second quarter 2009.

Bond data:

Nominal yields: continuously compounded U.S. government bond yields of maturities 1,2,3,4,5,6,7,8,9,10 years. Data is from first quarter 1972 to second quarter 2009.

Real yields: continuously compounded yields from U.S. Treasury Inflation Protected Securities (TIPS) with maturities of 5,6,7,8,9,10 years. Data is from first quarter 2003 to second quarter 2009.

All bond data is from the Board of Governors of the Federal Reserve System.

Volatility data:

Realized volatility of changes in nominal yields: I use daily squared differences in continuously compounded U.S. government bond yields of maturities 1,2,3,4,5,6,7,8,9,10 years to construct a quarterly measure of realized volatility. Data is from first quarter 1972 to second quarter 2009.

B. Proof of Proposition 1

Rewrite the constrained minimization in (12) as a relative entropy constrained HJB. J denotes the value function. It depends on $J = J(\ln c, \eta^w, \eta^r, z_t)$. The time varying Lagrange multipliers for the entropy constraints are θ_t^r and

θ_t^w .

$$\begin{aligned} \rho J(\ln c_t, \eta^w, \eta^r, z_t) &= \min_{h_t^r, h_t^w} \ln c_t + \theta_t^r \left(\frac{(h_t^r)^2}{2} - A^r (\eta_t^r)^2 \right) + \theta_t^w \left(\frac{(h_t^w)^2}{2} - A^w (\eta_t^w)^2 \right) + \\ &+ \mathcal{A}^h J(\ln c_t, \eta^w, \eta^r, z_t), \end{aligned} \quad (34)$$

where \mathcal{A}^h is the second order differential operator (under the ambiguity adjusted measure) applied to the value function J . Guess the value function is linear in the states, i.e. $J = \delta_0 + \delta_z z_t + \delta_{\eta^w} \eta_t^w + \delta_{\eta^r} \eta_t^r$. The second order differential operator applied to the value function is

$$\mathcal{A}^h J = \delta_z (\kappa_z + \sigma_{1z} h_t^r + \sigma_{2z} h_t^w) + \delta_{\eta^r} (a_{\eta^r} + \kappa_{\eta^r} \eta_t^r) + \delta_{\eta^w} (a_{\eta^w} + \kappa_{\eta^w} \eta_t^w) \quad (35)$$

First-order conditions with regard to h_t^r and θ_t^r reveal

$$\theta_t^r = \frac{-\sigma_{1z} \delta_z}{\pm \sqrt{2A^r} \eta_t^r} \quad (36)$$

$$h_t^r = \pm \sqrt{2A^r} \eta_t^r. \quad (37)$$

Note ($\delta_z > 0, \sigma_{1z} > 0$), the robust HJB is minimized at

$$h_t^r = -\sqrt{2A^r} \eta_t^r \equiv m^r \eta_t^r, \quad m^r \in \mathcal{R}^- \quad (38)$$

$$\theta_t^r = \frac{-\sigma_{1z} \delta_z}{-\sqrt{2A^r} \eta_t^r} \equiv \frac{b_0}{\eta_t^r}, \quad b_0 \in \mathcal{R}^+, \quad (39)$$

where we defined $m^r \equiv -\sqrt{2A^r} < 0$ and $b_0 \equiv \frac{-\sigma_{1z} \delta_z}{-\sqrt{2A^r}} > 0$. This proves the first part of the proposition.

First-order conditions with regard to h_t^w and θ_t^w reveal

$$\theta_t^w = \frac{-\sigma_{2z} \delta_z}{\pm \sqrt{2A^w} \eta_t^w} \quad (40)$$

$$h_t^w = \pm \sqrt{2A^w} \eta_t^w. \quad (41)$$

Note ($\delta_z > 0, \sigma_{2z} < 0$), the robust HJB is minimized at

$$h_t^w = \sqrt{2A^w} \eta_t^w \equiv m_w \eta_t^w, \quad m_w \in \mathcal{R}^+ \quad (42)$$

$$\theta_t^w = -\frac{\sigma_{2z} \delta_z}{\sqrt{2A^w} \eta_t^w} \equiv \frac{b_1}{\eta_t^w}, \quad b_1 \in \mathcal{R}^+, \quad (43)$$

where we defined $m_w \equiv \sqrt{2A^w} < 0$ and $b_1 \equiv \frac{-\sigma_{2z} \delta_z}{\sqrt{2A^w}} > 0$. This proves the remaining part of the proposition. Plug the solution to the robust HJB and verify that the guess of the linearized value function was correct.

C. Derivation of Real Bond Yields

Let $F^r = F_t^r(\tau)$ be the price of a real bond. In the economy this price is exponentially affine $F^r = e^{A^r(\tau) + B^r(\tau)S}$ where S denotes the state vector $S_t = (z_t h_t^r h_t^w)'$. F^r solves the following PDE

$$r \cdot F^r = \mathcal{A}^H F^r + F_t^r, \quad F_t^r = -F_\tau^r \quad (44)$$

where r is the real risk free rate, \mathcal{A}^H is the second order differential operator and F_τ^r is the first derivative of F^r with regard to τ . Using the equilibrium real interest rate and the exogenous dynamics of S reveals that the bond loadings solve simple ordinary differential equations. The solution for $B_z^r(\tau)$ is $B_z^r(\tau) = \frac{1}{\kappa^z}(1 - e^{\kappa^z \tau})$. The corresponding ode for B_{hw}^r and B_{hr}^r solve

$$\frac{d}{d\tau} B_{hw}^r(\tau) = \kappa_{\eta^w} B_{hw}^r(\tau) + \frac{1}{2} \sigma_{\eta^w}^2 m^w (B_{hw}^r(\tau))^2 + \sigma_{2z} B_z^r(\tau), \quad B_{hw}^r(0) = 0 \quad (45)$$

$$\frac{d}{d\tau} B_{hr}^r(\tau) = \kappa_{\eta^r} B_{hr}^r(\tau) + \frac{1}{2} \sigma_{\eta^r}^2 m^r (B_{hr}^r(\tau))^2 + \sigma_{1z} B_z^r(\tau), \quad B_{hr}^r(0) = 0 \quad (46)$$

$$(47)$$

The analytic solution to the Riccati equations (approximating $B_z^r(\tau)$ at its steady state value $B_z^r(\infty)$) is

$$B_{hr}^r(\tau) = \frac{(-\beta_1 + d)(1 - e^{d\tau})}{2\beta_2(1 - ge^{d\tau})} \quad (48)$$

$$g := \frac{-\beta_1 + d}{-\beta_1 - d} \quad (49)$$

$$d := \sqrt{\beta_1^2 - 4\beta_0\beta_2} \quad (50)$$

$$\beta_0 := \frac{\sigma_{1z}}{\kappa_z} \quad (51)$$

$$\beta_1 := \kappa_{\eta^r} \quad (52)$$

$$\beta_2 := 0.5m^r\sigma_{\eta^r}^2 \quad (53)$$

and

$$B_{hw}^r(\tau) = \frac{(-\beta_1 + d)(1 - e^{d\tau})}{2\beta_2(1 - ge^{d\tau})} \quad (54)$$

$$g := \frac{-\beta_1 + d}{-\beta_1 - d} \quad (55)$$

$$d := \sqrt{\beta_1^2 - 4\beta_0\beta_2} \quad (56)$$

$$\beta_0 := \frac{\sigma_{2z}}{\kappa_z} \quad (57)$$

$$\beta_1 := \kappa_{\eta^w} \quad (58)$$

$$\beta_2 := 0.5m^w\sigma_{\eta^w}^2. \quad (59)$$

Function $A^r(\tau)$ follows from direct integration.

D. Derivation of Nominal Bond Yields

Let $F = F_t(\tau)$ be the price of a nominal bond. In the economy this price is exponentially affine $F = e^{A^n(\tau) + B^n(\tau)X}$ where X denotes the state vector $X_t = (w_t S_t)'$. F solves the following PDE

$$R \cdot F = \mathcal{A}^H F + F_t, \quad F_t = -F_\tau \quad (60)$$

where R is the nominal short rate, \mathcal{A}^H is the second order differential operator and F_τ is the first derivative of F with regard to τ . Using the equilibrium

nominal short rate and the exogenous dynamics of X reveals that the bond loadings solve simple ordinary differential equations. The solution to the loadings is as follows: $B_z^n(\tau) = B_z^r(\tau)$, $B_{h^r}^n(\tau) = B_{h^r}^r(\tau)$, $B_w^n(\tau) = \frac{1}{\kappa_w}(1 - e^{\kappa_w \tau})$. The analytical solution to $B_{h^w}^n(\tau)$ (approximated $B_w^n(\tau)$ at its steady state value $B_w^n(\tau)$) is

$$B_{h^w}^n(\tau) = \frac{(-\beta_1 + d)(1 - e^{d\tau})}{2\beta_2(1 - ge^{d\tau})} \quad (61)$$

$$g := \frac{-\beta_1 + d}{-\beta_1 - d} \quad (62)$$

$$d := \sqrt{\beta_1^2 - 4\beta_0\beta_2} \quad (63)$$

$$\beta_0 := \frac{\sigma_{2z}}{\kappa_z} + \frac{\sigma_w}{\kappa_w} \quad (64)$$

$$\beta_1 := \kappa_{\eta^w} \quad (65)$$

$$\beta_2 := 0.5m^w\sigma_{\eta^w}^2. \quad (66)$$

Function $A^n(\tau)$ follows from direct integration.

E. Derivation of Detection Error Probability

The derivation of the detection-error probabilities $p_T(m^r, m^w)$ follows directly from Maenhout (2006):

$$p_T(m^r, m^w) = \frac{1}{2} \left(Pr \left(\ln \frac{dQ_T^h}{dQ_T^0} > 0 | dQ^0, \mathcal{F}_0 \right) + Pr \left(\ln \frac{dQ_T^0}{dQ_T^h} > 0 | dQ^h, \mathcal{F}_0 \right) \right) \quad (67)$$

$$\begin{aligned} &= \frac{1}{2} \left(Pr \left(-\frac{1}{2} \int_0^T h'_m h_m dm + \int_0^T h_m \cdot dW_m^z > 0 | dQ^0, \mathcal{F}_0 \right) \right) \\ &+ \frac{1}{2} \left(Pr \left(-\frac{1}{2} \int_0^T h'_m h_m dm - \int_0^T h_m \cdot dW_m^{z,h} > 0 | dQ^h, \mathcal{F}_0 \right) \right) \end{aligned} \quad (68)$$

where $h_t = (m^w \eta_t^w \ m^r \eta_t^r)'$ is the endogenous distortion to trend GDP growth. The last equation coincides with

$$p_T(m^r, m^w) = \frac{1}{2} - \frac{1}{2\pi} \int_0^\infty \left(\operatorname{Re} \left(\frac{\phi^h(k, 0, T)}{ik} \right) - \operatorname{Re} \left(\frac{\phi(k, 0, T)}{ik} \right) \right) dk \quad (69)$$

where $i = \sqrt{-1}$, $\phi(\cdot)$ is defined as $\phi(k, 0, T) := E [e^{i \cdot k \cdot \xi_{1,T}} | \mathcal{F}_0]$ and $\phi^h(\cdot)$ is defined as $\phi^h(k, 0, T) := E^h [e^{i \cdot k \cdot \xi_{1,T}} | \mathcal{F}_0]$ and $\xi_{1,T} = \ln \frac{dQ_T^h}{dQ_T^0}$.

Applying Feynman-Kac theorem to ϕ^h and ϕ reveals that they are an exponentially quadratic function in the amount of inflation distortion h_t :

$$\phi^h(k, t, T) = z_t^{ik+1} e^{G(\tau, k) + \sum_{j \in \{w, r\}} E_j(\tau, k) h_j(t) + \sum_{j \in \{w, r\}} \frac{F_j(\tau, k)}{2} h_j^2(t)} \quad (70)$$

$$\phi(k, t, T) = z_t^{ik} e^{\hat{G}(\tau, k) + \sum_{j \in \{w, r\}} \hat{E}_j(\tau, k) h_j(t) + \sum_{j \in \{w, r\}} \frac{\hat{F}_j(\tau, k)}{2} h_j^2(t)} \quad (71)$$

$$z_T := e^{\xi_{1,T}}, \quad (72)$$

where $G(\tau, k), E_j(\tau, k), F_j(\tau, k), \hat{G}(\tau, k), \hat{E}_j(\tau, k), \hat{F}_j(\tau, k)$ are deterministic solutions to standard complex valued Riccati equations. I provide details on the derivation of the Riccati equations for ϕ^h . The derivation of ϕ is analogous.

In order to get an analytical solution, I approximate the conditional volatility of the uncertainty premium by its steady state value, i.e.

$$dh_t = (a_\eta m + \kappa_\eta h_t) dt + \sigma_\eta \sqrt{m} \sqrt{h_t} dW_t^\eta \quad (73)$$

$$\approx (a_\eta m + \kappa_\eta h_t) dt + \sigma_\eta \sqrt{m} \sqrt{m a_\eta / (-\kappa_\eta)} dW_t^\eta \quad (74)$$

For ease of notation I define $b := \sqrt{m} \sqrt{m a_\eta / (-\kappa_\eta)}$, where more specifically b_r refers to the conditional steady state volatility of dh^r and b_w is the analog for dh^w .

$\phi^h(k, t, T)$ solves $\phi_\tau^h = \mathcal{A}\phi^h$ where $\tau = T - t$ and ϕ_τ^h stands for $\frac{\partial \phi^h}{\partial \tau}$.

$$\phi_\tau^h = \phi^h \left(\dot{G}(\tau, k) + \sum_{j \in \{w, r\}} \dot{E}_j(\tau, k) h_j(t) + \frac{1}{2} \sum_{j \in \{w, r\}} \dot{F}_j(\tau, k) h_j^2(t) \right) \quad (75)$$

$$\begin{aligned} \frac{\mathcal{A}\phi^h}{\phi^h} &= \sum_{j \in \{w, r\}} [(E_j(\tau, k) + F_j(\tau, k) h_j(t)) (a_{\eta^j} m^j + \kappa_{\eta^j} h_j(t))] + 0.5ik(k+1) (h_w^2(t) + h_r^2(t)) \\ &+ \frac{1}{2} \sum_{j \in \{w, r\}} (E_j^2(\tau, k) + F_j^2(\tau, k) h_j^2(t) + 2E_j(\tau, k) F_j(\tau, k) h_j(t)) b_j^2 \end{aligned} \quad (76)$$

Set $\phi_\tau^h = \mathcal{A}\phi^h$ and match coefficients:

$$F_j(\tau, k) = F_j^r(\tau, k) + F_j^c(\tau, k) \quad (77)$$

$$F_j^c(\tau, k) = k \cdot \tau \quad (78)$$

$$F_j^r(\tau, k) = \frac{(a_j + d_j)(1 - e^{d_j \tau})}{2b_{2j}^r(1 - g_j e^{d_j \tau})} \quad (79)$$

where F^r is the real part of F and F^c is the complex part and

$$a_j = -b_{1j}^r; \quad d_j = \sqrt{a_j^2 - 4b_{0j}^r b_{2j}^r}; \quad g_j = \frac{a_j + d_j}{a_j - d_j}; \quad b_{0j}^r = -k^2 \quad (80)$$

$$b_{1j}^r = 2\kappa_{\eta^j}; \quad b_{2j}^r = b_j^2 \quad (81)$$

where $j \in \{w, r\}$. The stable steady state solution of F is

$$F_j(\infty, k) = -\frac{b_{1j}^r + d_j}{2b_{2j}^r}. \quad (82)$$

The loadings $E_j(\tau, k)$, $j \in \{f, w, r\}$ solve the following ode

$$\frac{d}{d\tau} E_j(\tau, k) = \kappa_{\eta^j} E_j(\tau, k) + m_j a_{\eta^j} F_j(\tau, k) + E_j(\tau, k) F_j(\tau, k) b_j^2. \quad (83)$$

We obtain an analytical approximation by approximating $F_j(\tau, k)$ around its

steady state value $F_j(\infty, k)$.

$$E_j(\tau, k) = -\frac{\hat{a}_j}{\hat{b}_j}(1 - e^{\hat{b}_j\tau}) \quad (84)$$

$$\hat{a}_j = F_j(\infty, k)m_j a_{\eta^j} \quad (85)$$

$$\hat{b}_j = F_j(\infty, k)b_j^2 + \kappa_{\eta^j}. \quad (86)$$

The loading $G(\tau, k)$ is obtained through straightforward integration

$$G(\tau, k) = \sum_{j \in \{f, w, r\}} \left(m_j a_{\eta^j} \int_0^\tau E_j(u, k) du \right) + \frac{1}{2} \sum_{j \in \{w, r\}} b_j^2 \int_0^\tau E_j^2(u, k) du. \quad (87)$$

The required expression $\phi^h(k, 0, T)$ is therefore

$$\phi^h(k, 0, T) = e^{G(T, k) + \sum_{j \in \{w, r\}} E_j(T, k) h_j(\infty) + \frac{1}{2} \sum_{j \in \{w, r\}} F_j(T, k) h_j^2(\infty)}, \quad (88)$$

where we assumed that $h_j(0)$ started in its steady state $h_j(\infty) = \frac{m_j a_{\eta^j}}{-\kappa_{\eta^j}}$.

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Table 1: PARAMETER ESTIMATES (Standard Errors)

Panel A: State Variables

Drift, Volatility			
	κ	σ	a
w	-0.049 (0.0004)	0.0023 (0.00002)	0 (fixed)
z	-0.27 (0.002)	0.0078 (0.00039)	-0.0018 (0.00005) 0 (fixed)
η^w	-0.49 (0.007)	0.78 (0.16)	0.0015 (0.00025)
η^z	-0.22 (0.002)	0.50 (0.95)	0.0017(0.00075)

Panel B: Growth and Inflation

c_0	0.0065 (fixed)
p_0	0.0096 (fixed)
σ_c	8.8e-5 (4e-5)
σ_p	5.4e-5 (4e-5)
ρ	0.001 (fixed)
m^w	40.97 (0.496)
m^r	-0.93 (0.042)

Note: The table presents ML parameter estimates and their standard error (in parenthesis). The asymptotic standard errors are determined based on the score of the log likelihood. The ML estimation uses bond yield, bond volatility and macro data from 1972.I to 2009.II.

Table 2: Yield Curve, in %, per quarter

Panel A: Nominal Yields

y^n											
maturity	R	4	8	12	16	20	24	28	32	36	40
data	1.6	1.5822	1.6391	1.6819	1.7159	1.7453	1.7717	1.795	1.8154	1.8327	1.8484
model	1.5991	1.5577	1.5915	1.6354	1.6797	1.7215	1.7601	1.7952	1.8270	1.8558	1.8817

Panel B: Real Yields

y^r							
maturity	20	24	28	32	36	40	
data	0.3782	0.4109	0.4400	0.4650	0.4859	0.5014	
model	0.5274	0.5233	0.5204	0.5181	0.5164	0.5150	

Panel C: Volatility of Nominal Yields

maturity	4	8	12	16	20	24	28	32	36	40
data	0.1221	0.1321	0.1344	0.1326	0.1298	0.1269	0.1245	0.1226	0.1214	0.1206
model	0.4050	0.2060	0.1385	0.1041	0.0834	0.0695	0.0596	0.0521	0.0464	0.0417

Note: Panel A compares model implied nominal bond yields with the data counterpart. R stands for the nominal short rate (federal funds rate), while the other maturities refer to quarters. The yields are in quarterly percentage units. Panel B compares model implied real bond yields with the data counterpart. Maturity is in quarterly units, and interest rates are in quarterly percentage units. Panel C compares the model implied volatility of nominal yields with the data counterpart. Volatilities are in quarterly units and in percent. The MLE estimation uses bond yield, bond volatility and macro data from 1972.I to 2009.II.

Table 3: Variance Decomposition

Panel A: Nominal Yields

<i>yⁿ</i>					
maturity	total	<i>w</i>	<i>z</i>	<i>h^w</i>	<i>h^r</i>
4	1.0000	0.7358	0.2637	0.0004	0.0000
8	1.0000	0.8397	0.1602	0.0001	0.0000
12	1.0000	0.8937	0.1062	0.0001	0.0000
16	1.0000	0.9228	0.0771	0.0001	0.0000
20	1.0000	0.9395	0.0604	0.0000	0.0000
24	1.0000	0.9498	0.0502	0.0000	0.0000
28	1.0000	0.9565	0.0435	0.0000	0.0000
32	1.0000	0.9611	0.0389	0.0000	0.0000
36	1.0000	0.9643	0.0356	0.0000	0.0000
40	1.0000	0.9667	0.0333	0.0000	0.0000

Panel B: Annual Bond Premiums

<i>BondPremium</i>					
maturity	total	<i>w</i>	<i>z</i>	<i>h^w</i>	<i>h^r</i>
4	1.0000	0	0	0.9313	0.0687
8	1.0000	0	0	0.9769	0.0231
12	1.0000	0	0	0.9886	0.0114
16	1.0000	0	0	0.9929	0.0071
20	1.0000	0	0	0.9950	0.0050
24	1.0000	0	0	0.9961	0.0039
28	1.0000	0	0	0.9967	0.0033
32	1.0000	0	0	0.9972	0.0028
36	1.0000	0	0	0.9975	0.0025
40	1.0000	0	0	0.9977	0.0023

Note: This table summarizes several variance decompositions. Panel A depicts variance decompositions for nominal bond yields. Panel B presents variance decompositions for the 4-quarter model implied expected excess return of holding a nominal bond with different maturities. The ML estimation uses bond yield, bond volatility and macro data from 1972.I to 2009.II.

Table 4: Variance Decomposition

Panel A: Implied Volatility

Black Implied Volatilities

maturity	total	w	z	h^w	h^r
4	1.0000	0	0	0.9982	0.0018
8	1.0000	0	0	0.9933	0.0067
12	1.0000	0	0	0.9901	0.0099
16	1.0000	0	0	0.9886	0.0114
20	1.0000	0	0	0.9880	0.0120
24	1.0000	0	0	0.9878	0.0122
28	1.0000	0	0	0.9878	0.0122
32	1.0000	0	0	0.9877	0.0123
36	1.0000	0	0	0.9878	0.0122
40	1.0000	0	0	0.9877	0.0123

Note: This table shows variance decompositions for the Black implied volatility for a three month call on several interest rates. The MLE uses bond yield, bond volatility and macro data from 1972.I to 2009.II.

Figure 1: **Observable Long-Run Risk and Long-Run Ambiguity**

This figure shows the four state variables of the economy. The upper panel plots the predictable components in GDP growth and in inflation, z and w , respectively. The lower panel plots the amount of model misspecification doubts about the data generating process that drives z and w . All states are observed in real-time, annualized and in %. The sample is from 1972.I to 2009.II. The data is from the Survey of Professional Forecasters.

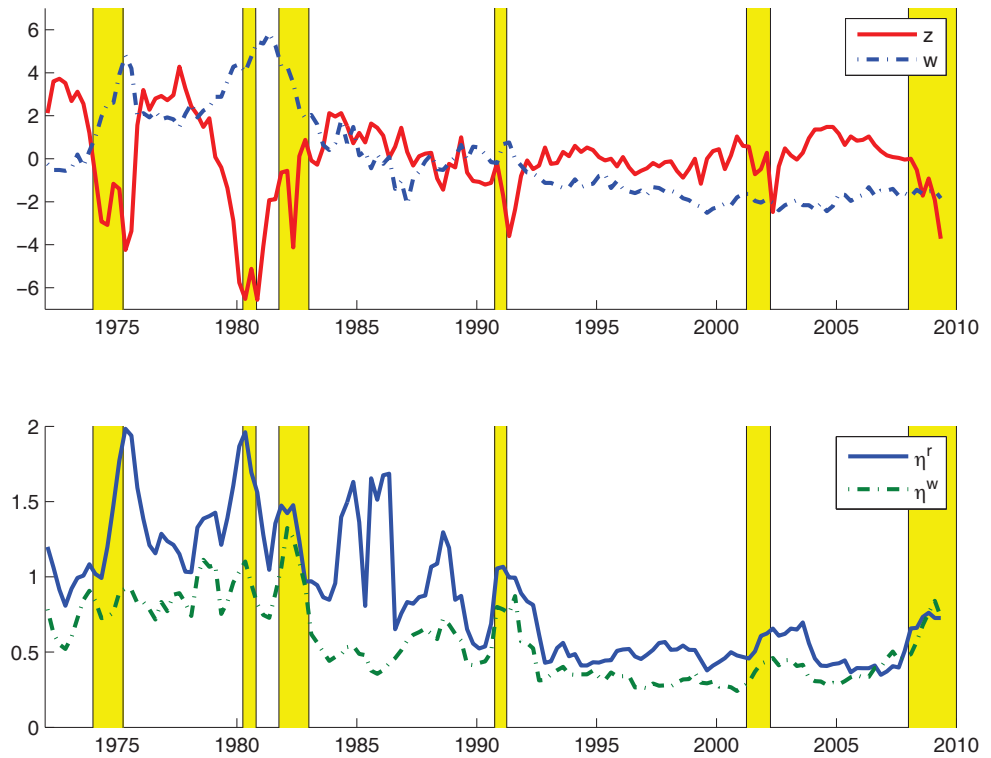


Figure 2: **Impulse Responses: Yield Curve**

This figure presents impulse responses of the yield curve for a one percent increase in η^r and η^w . The x-axis is in quarters, the y-axis is in percent. The model is estimated with ML and uses data from 1972.I to 2009.II.

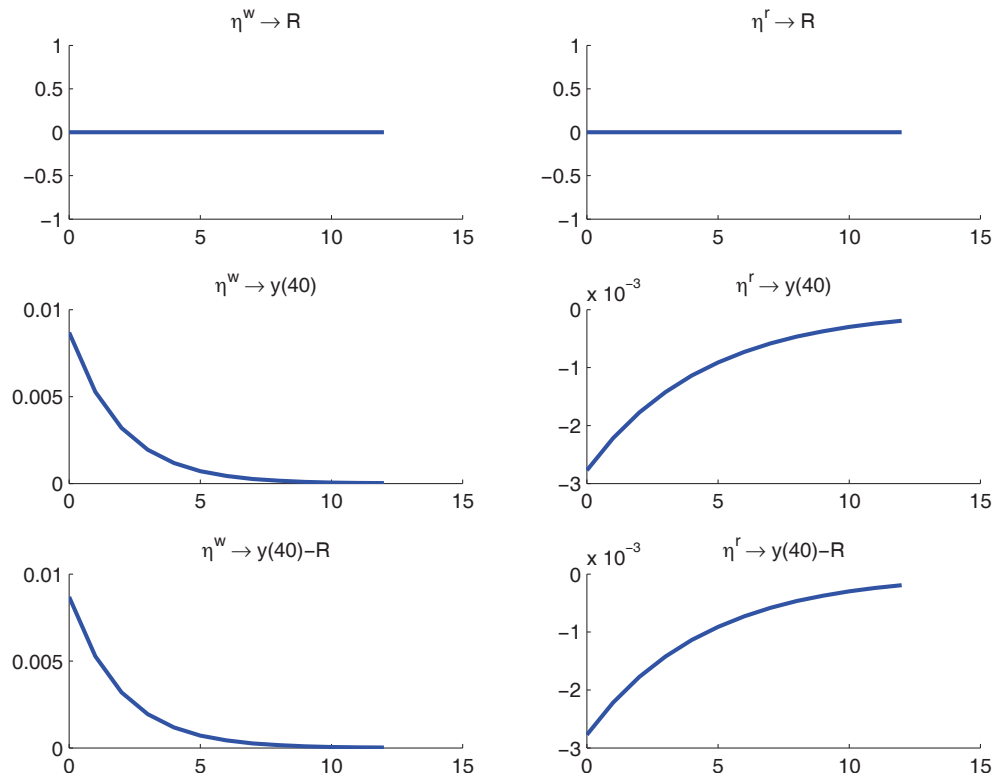


Figure 3: **Impulse Responses: Bond Premium**

This figure presents impulse responses for a one percent increase in η^r (lower panel) and η^w (upper panel) on itself and on bond premiums. The x-axis is in quarterly units, while the y-axis is in percentage units. The upper panel, from left to right, shows the response to the bond premium of a 4-quarter and a 40-quarter bond. These premiums are denoted Bond Premium(4) and Bond Premium(40). The model is estimated with yield and macro data and uses a one step MLE with data from 1972.I to 2009.II.

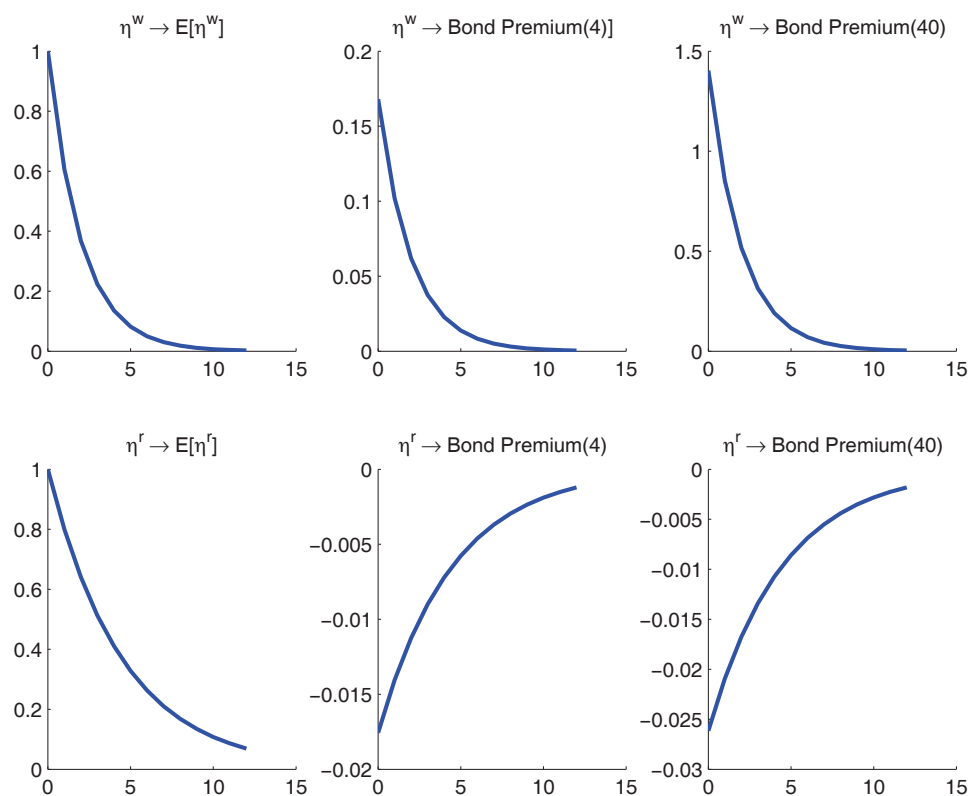


Figure 4: **Term Structure of Bond Premiums**

This figure presents the cross-section of bond premiums in the data and in the model. The bond premium is calculated as the 4-quarter holding period return of all nominal bonds minus the nominal short rate (federal funds rate). The x-axis is in quarterly units and corresponds to the maturity of the nominal bond. The y-axis is in percent. The premiums are annualized. The dotted line corresponds to the realized sample average. The solid line stands for the model implied expected excess bond return. The $-.$ line stands for the empirical 3 standard deviation confidence interval around the realized sample average return. The confidence interval is determined as the realized sample average plus and minus three times the sample standard deviation divided by the square-root of the sample size. It represents the data estimate for the expected excess bond return in population. The model is estimated with yields and macro data in a one step MLE. The data length is 1972.I to 2009.II.

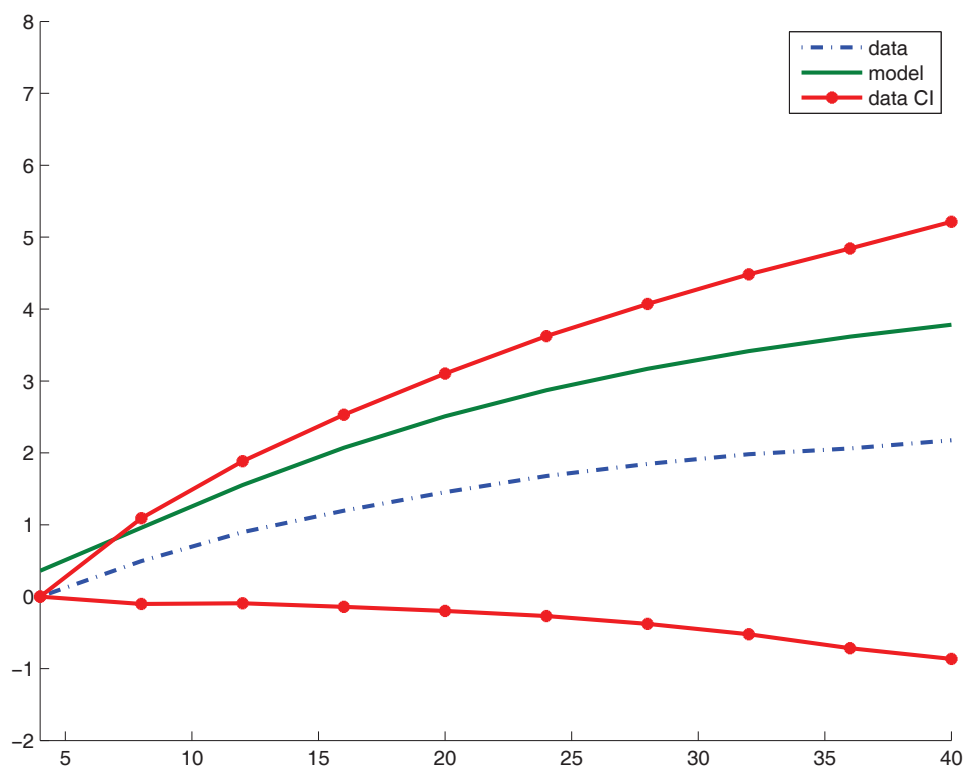


Figure 5: **Option Implied Volatilities**

This figure presents Black (1976) implied volatilities of a three month call on several Treasury bond yields. The panels from left to right represent the Black implied volatilities for the 8-quarter, 12-quarter, 16-quarter, and 20-quarter yield. The x-axis shows the moneyness, i.e. $\ln \frac{K}{y_t(\tau)}$. The y-axis is in percent. The ML estimates and sample mean of the states are used to construct the pictures. The model is estimated with yield and macro data and uses a one step ML estimation with data from 1972.I to 2009.II.

